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ELECTROMAGNETIC THEORY AND THE FOUNDATIONS OF ELECTRIC CIRCUIT THEORY

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A MATHEMATICAL ANALYSIS OF THE FOUNDATIONS OF THE EQUATIONS OF ELECTRIC CIRCUIT THEORY IN WHICH THEY ARE DERIVED DIRECTLY FROM MAXWELL'S EQUATIONS AND THEIR LIMITATIONS POINTED OUT

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Electromagnetic Theory and the Foundations of Electric Circuit Theory [1]

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SYNOPSIS: The familiar equations which are used to solve for the currents and charges in linear networks summarize the inductive analysis of countless observations made upon such networks. Having been arrived at by inductive methods, these familiar equations of Ohm, Faraday and Kirchhoff are substantially independent of the more general electromagnetic theory of Maxwell and Lorentz. The present paper examines the foundation of electric circuit theory from the standpoint of the fundamental equations of electromagnetic theory and a derivation of the former from the latter is made, in the course of which the assumptions, approximations and restrictions tacitly involved in the equations of circuit theory are explicitly stated. The treatment is sufficiently extended as to show how the familiar equation for the simple oscillating circuit and the so-called telegraph equation can be deduced from the Maxwell-Lorentz statement of electromagnetic theory.

ELECTRIC circuit theory, as the term is employed in the present paper, is that branch of electromagnetic theory which deals with electrical oscillations in linear networks; more precisely stated, with the distribution of currents and charges in the free oscillations of the network, or under the action of impressed electromotive forces. The network is a connected set of closed circuits or meshes each of which is regarded as made up of inductances, resistances and condensers, a simplifying assumption which is fundamental to circuit theory.

The great importance of electric circuit theory in electro-technics does not require emphasis; it is not too much to say that it is responsible in no small measure for the rapid development of electrical engineering and is an absolutely essential guide in the complicated technical problems there encountered.

The equations of electric circuit theory in their present form are essentially a generalization of the observations of Ohm, Faraday, Henry, Kirchhoff and others and their development preceded the electromagnetic theory of Maxwell and Lorentz. Naturally, in view of its early development, circuit theory embodies approximations, the precision of which cannot be determined from the observations on which it is based. For example, circuit theory explicitly ignores the finite velocity of propagation of electromagnetic disturbances, and

[1] In its original form this paper was read before the National Academy of Sciences, April 1925. Subsequently it was amplified and revised and included in a lecture course delivered at the Massachusetts Institute of Technology, April 1926.

hence the phenomena of radiation. Again it involves the assumption that the network can be represented by a finite number of coordinates and thus that it constitutes a rigid dynamic system. The rigorous equations of electromagnetic theory formulate the relations between current and charge *densities* and the accompanying fields. Circuit theory, on the other hand, expresses approximate relations between total currents and charges and impressed electromotive forces.

With the rapid development of electro-technics an increasing number of problems is being encountered where the application of classical electric circuit theory is of doubtful validity, or where the conclusions derived from it must be interpreted with great care. Such problems are the result not only of the use of very high frequency in radio-transmission but arise also in connection with the need of a more precise theory of wire transmission.

In view of the foregoing it seems desirable to examine the foundations of circuit theory. This is the problem dealt with in the present paper:—a derivation of the classical circuit theory equations from the standpoint of electromagnetic theory, in the course of which the approximations involved are pointed out.

A second reason, pedagogic in character, is believed to justify the present study. This is, that, as circuit theory is usually taught to technical students no picture of its true relation to electromagnetic theory is given, and the student comes to regard inductance, resistance, capacity, voltage, etc., as fundamental concepts.

To start with our problem in a general form, consider a conducting system of any form whatsoever, in which the *charge density* at any point x, y, z at any time t is denoted by

$$\rho(x, y, z, t) = \rho,$$

and the *vector current density* by

$$u(x, y, z, t) = u,$$

the functional notation indicating that the charge and the current density are functions of space and time. At any point in the system let

$$E(x, y, z, t) = E$$

denote the *vector electric intensity*. This we shall suppose to be composed of two parts; thus

$$E = E^\circ + E'. \tag{1}$$

In this equation E° is the *impressed electric intensity* and E' the *electric intensity due to the reaction of the currents and charges in the system*. Thus E° may be the electric intensity due to a distant system, as in radio transmission, or that due to a generator, battery or other

source of energy. In the following we shall suppose that E° is specified and we shall keep carefully in mind the fact that E° denotes the electric intensity *not* due to the reaction of the system itself. This distinction is extremely important.

We have now to take up the problem of specifying the electric intensity E' in terms of the currents and charges of the system. The necessary relation is furnished by the *Lorentz or retarded potentials*

$$\Phi = \int \frac{\rho(t - r/c)}{r} dv, \quad \text{(scalar)} \tag{2}$$

$$A = \int \frac{u(t - r/c)}{r} dv, \quad \text{(vector).} \tag{3}$$

Interpreting equation (2), Φ is equal to the volume integral of the *retarded* charge density divided by the distance between the point at which Φ is evaluated and the location of the charge. The *retarded* charge density means that at time t we take the value of the charge at the earlier time $t - r/c$, where c is the velocity of light. It is to be understood that ρ and u are the true charge and current density, and displacement currents are not included. Their effect appears in the retardation only. c also is the true velocity of propagation in vacuo. The potential Φ is therefore a generalization of the electrostatic potential into which it degenerates in an unvarying system.

Similarly the *vector potential* A of equation (3) is gotten by a volume integral of the retarded vector current density divided by distance r. As the name indicates it is a vector quantity and in Cartesian coordinates has three components A_x, A_y, A_z.

By means of the equation

$$E' = -\operatorname{grad} \Phi - \frac{1}{c}\frac{\partial}{\partial t} A, \tag{4}$$

the electric intensity due to the reaction of the system is expressed in terms of the charge and current densities.

Equations (2), (3), (4) and the additional equations

$$B' = \operatorname{curl} A, \tag{5}$$

$$\operatorname{div} u = -\frac{1}{c}\frac{\partial}{\partial t} \rho, \tag{6}$$

(where B' is the magnetic induction due to the currents in the system) are the complete equivalent of Maxwell's equations from which they are immediately derivable.

Aside from the fact that the physical significance of the foregoing equations is deducible by direct inspection, they represent a great step because they are *integral equations* whereas Maxwell's equations are *partial differential equations*. A second advantage is that only *true* currents and charges are involved, the displacement currents of Maxwell being replaced by *retarded action at a distance*. Whatever may be said for or against the physical point of view, this effects a substantial mathematical simplification. The formulation of the fundamental field equations in terms of the retarded potentials is due to Lorentz.

In order to complete the specification of the system we have to formulate the relation between the current density $\boldsymbol{u}$ and the electric intensity $\boldsymbol{E}$. In doing so we shall exclude magnetic matter and shall assume that the conductors obey Ohm's law. This restriction is not necessary but effects a great simplification in both the physical picture and the mathematical formulas.[2] We therefore assume that the conducting system is specified completely by its conductivity

$$g = g(x, y, z),$$

and that

$$\frac{1}{g}\boldsymbol{u} = \boldsymbol{E}. \tag{7}$$

Combining with (1) and (4), we have

$$\frac{1}{g}\boldsymbol{u} = \boldsymbol{E}^\circ - \operatorname{grad}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}, \tag{8}$$

which is our fundamental equation.[3] The preceding set of equations, if g and $\boldsymbol{E}^\circ$ are everywhere specified, enable us, theoretically at least, to completely solve the problem of the distribution of currents and charges in the system.

Before taking up this problem we shall first derive the energy theorem and then investigate the properties of the field by aid of the retarded potentials.

Starting with equation (8), multiply throughout by $\boldsymbol{u}$, getting

$$\frac{1}{g}u^2 = (\boldsymbol{E}^\circ\cdot\boldsymbol{u}) - (\boldsymbol{u}\cdot\operatorname{grad}\Phi) - \left(\boldsymbol{u}\cdot\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}\right),$$

and integrate over the system, getting

$$\int\frac{1}{g}u^2dv = \int(\boldsymbol{E}^\circ\cdot\boldsymbol{u})dv - \int(\boldsymbol{u}\cdot\operatorname{grad}\Phi)dv - \int\left(\boldsymbol{u}\cdot\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}\right)dv.$$

[2] See Appendix for the general formulas.

[3] See Appendix for the vector notation employed in this paper.

Remembering that u is expressed in *elm.* units, this becomes

$$\frac{1}{c}D = \frac{1}{c}W - \int (u\cdot\text{grad }\Phi)dv - \int \left(u\cdot\frac{1}{c}\frac{\partial}{\partial t}A\right)dv$$

or

$$W = D + c\int (u\cdot\text{grad }\Phi)dv + c\int\left(u\cdot\frac{1}{c}\frac{\partial}{\partial t}A\right)dv,$$

where W is the work done per unit time by the impressed electric field, and D is the *dissipation* per unit time in the system; i.e., the rate at which electrical energy is converted into heat. By means of general theorems in vector analysis, the integrals can be transformed and the equation reduced to the form

$$W = D + \frac{\partial}{\partial t}\frac{1}{8\pi}\int (E^2 + H^2)dv + \frac{c}{4\pi}\int [E\cdot H]_n dS,$$

the last integral being taken over any closed surface which includes the system. Translating this equation into words, it states that:—

The work done per unit time by the impressed forces is equal to the rate of dissipation per unit time plus the rate of increase of the field energy plus the rate at which energy is *radiated from* the system. The vector $(c/4\pi)[E\cdot H]$ is the *radiation vector* and gives the density and direction of energy flow per unit time;[4] it will be denoted by S.

We now shall briefly consider the field due to the currents and charges in the system.

If the current density u and charge density ρ are everywhere specified, the retarded potentials are uniquely and completely determined by the formulas

$$A = \int \frac{u(t - r/c)}{r}dv, \qquad \text{(vector)}$$

$$\Phi = \int \frac{\rho(t - r/c)}{r}dv. \qquad \text{(scalar)}.$$

The functional notation $u(t - r/c)$ and $\rho(t - r/c)$ indicating that u and ρ are to be evaluated at time $t - r/c$ may profitably be replaced by $ue^{-(p/c)r}$ and $\rho e^{-(p/c)r}$, so that

$$A = \int \frac{ue^{-(p/c)r}}{r}dv,$$

$$\Phi = \int \frac{\rho e^{-(p/c)r}}{r}dv.$$

[4] This is known as Poynting's theorem.

These expressions may be interpreted in either of two ways. (1) If $p = i\omega$ where $\omega = 2\pi f$ and $i = \sqrt{-1}$, then the formulas are the usual complex steady state expressions. On the other hand if p is regarded as d/dt, they are *operational formulas*. It is worth while to explain the latter briefly on account of its own interest and its bearing on the operational calculus.

The differential equations of A and Φ are

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A = 4\pi u,$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \Phi = 4\pi \rho.$$

where

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2.$$

Now it will be recalled that the differential equation of the electrostatic potential V is

$$\nabla^2 V = 4\pi\rho$$

and that its solution is

$$V = \int \frac{\rho}{r} dv.$$

Operationally

$$\left(\nabla^2 - \frac{p^2}{c^2}\right)\Phi = 4\pi\rho$$

and the corresponding solution is

$$\Phi = \int \frac{\rho e^{-(p/c)}}{r} dv.$$

Now this is an operational equation in which ρ is an arbitrary time function. Its solution depends on the following general operational theorem.[5]

If x is defined by the operational equation

$$x = f(t)e^{-\lambda p},$$

then

$$x = f(t - \lambda).$$

Consequently, the solution of the operational equation for Φ is[6]

$$\Phi = \int \frac{\rho(t - r/c)}{r} dv.$$

[5] See "The Heaviside Operational Calculus," *Bull. Amer. Math. Soc.*, Jan., 1926.

[6] A proof of this theorem by operational methods was privately communicated to the author several years ago by Stuart Ballantine.

Let us now examine the field of the currents and charges by aid of the formulas

$$E = -\operatorname{grad}\Phi - \frac{p}{c}A,$$
$$H = \operatorname{curl} A.$$

Performing the indicated operations,

$$\operatorname{curl} A = -\int e^{-(p/c)r}[n\cdot u]\left(\frac{1}{r^2}+\frac{p}{c}\frac{1}{r}\right)dv,$$
$$\operatorname{grad}\Phi = -\int e^{-(p/c)r}\rho\cdot n\left(\frac{1}{r^2}+\frac{p}{c}\frac{1}{r}\right)dv,$$

where n is a unit vector, parallel to r, drawn through the contributing element.

We see from these formulas that the magnetic field due to the currents, and the electric field due to the charges, consist each of two components; one varying inversely as the square of the distance from the contributing element and the other inversely as the distance. Writing $p = i\omega = i\cdot 2\pi f$, the orders of magnitude of the two components are $1/r^2$ and ω/c^2 and their ratio is $2\pi(r/\lambda)$, where λ is the wave length.

The first component is the *induction field*, and involves the frequency only through the exponential term; the second is the *radiation* field and involves the frequency linearly.

If we are considering points in the system itself, and if the dimensions of the system are so small that $2\pi(r/\lambda)$ is small compared with unity, the expressions reduce to

$$\operatorname{curl} A = -\int \frac{[n\cdot u]}{r^2}dv,$$
$$\operatorname{grad}\Phi = -\int n\frac{\rho}{r^2}dv.$$

If therefore the dimensions of the system are sufficiently small with respect to the wave length, these expressions can be employed in calculating the distribution of the currents and charges in the system. This is usually the case in circuit theory, even at radio frequencies.

At a great distance from the system, however, the case is quite different. For no matter how large the wave length, λ, if we consider points outside the system such that $2\pi(r/\lambda)$ is everywhere large compared with unity, the second or radiation field will predominate. This leads to the important conclusion that the field which determines the distribution of currents and charges in the system is quite different

from the field which determines the radiation, and explains the fact that radiation may usually be neglected in calculating the distribution in the network.[7]

To examine the radiation field, consider a point P at such a distance from the system that $2\pi(r/\lambda)$ is very large. Choose any point in the system as the origin and write r_0 as the distance from the origin to the point P, and r' the distance from the contributing element P' to the origin. Then

$$r = r_0 - (r' \cdot n),$$

where n is the unit vector parallel to r_0 and

$$A = \frac{e^{-i\omega r_0}}{r_0} \int u \cdot e^{i\omega(r' \cdot n)} dv = \frac{e^{-i\omega r_0}}{r_0} J,$$

$$\text{curl } A = -i\omega \int \frac{e^{-i\omega r_0}}{r_0} [n \cdot J],$$

which determines H.

Instead of calculating E from the formula

$$-\text{grad } \Phi - \frac{i\omega}{c} A,$$

we make use of the fact that in the dielectric

$$i\omega E = \text{curl } H,$$

whence

$$E = -[H \cdot n].$$

The interpretation of these equations is that in the radiation field E and H are equal, are in phase and are perpendicular to each other and to the vector r_0. Consequently the radiation vector S is given by

$$S = \frac{c}{4\pi} H^2$$

$$= \frac{c\omega^2}{4\pi} \frac{|J|^2}{r^2},$$

and the radiation is everywhere outward.

These formulas can be used to calculate the radiation in terms of the current distribution alone, and the charge distribution does not appear explicitly.

[7] Conversely the field in the immediate neighborhood of the system is no criterion of the radiation field or the radiating properties of the system. This fact is not always kept in mind by radio-engineers.

Derivation of the Familiar Circuit Theory Relations

In the foregoing we have tacitly assumed that the distribution of currents and charges in the systems is known. We now take up the more difficult problem of determining the distribution in terms of the impressed field and the geometry and electrical constants of the system. This will introduce us to circuit theory and the enormous complexity of the general rigorous expressions will show its important role in physics and engineering. In fact without the beautiful simplifications of circuit theory very few problems of this type could be solved.

In taking up this problem there are two possible modes of approach. In accordance with one we start with Maxwell's differential equations and try to find a solution which satisfies the geometry of the system and the boundary conditions. For conducting systems of simple geometrical shapes solutions in this way are possible. For complicated networks, however, this mode of approach is quite hopeless.

The other mode of approach is to start with the equation

$$\begin{aligned} \frac{1}{g} u &= E^\circ - \operatorname{grad} \Phi - i\omega A \\ &= E^\circ - \operatorname{grad} \int \frac{\rho(t - r/c)}{r} dv - i\omega \int \frac{u(t - r/c)}{r} dv, \end{aligned} \tag{8}$$

which, together with the relation

$$i\omega\rho = - \operatorname{div} u,$$

is an integral equation which completely determines the distribution of currents and charges in the system provided g and E° are specified.

For general purposes of calculation it is quite hopeless as it stands. It has, however, several advantages. First, it is a direct and complete statement of the physical relations which obtain everywhere. Second, it uniquely determines the distribution and does not, like the differential equations, involve the determination of integration constants from the boundary conditions. Third, it leads, through appropriate approximations, to the philosophy and equations of circuit theory.

To start with a simple case, the solution of which can be extended without difficulty to the general network, consider a conductor forming a closed circuit. We suppose that it is exposed at every point to an impressed electric force E°, and we suppose that the surrounding dielectric is perfectly non-conducting. It is now our problem to derive, for this simple circuit, the circuit equations, in terms of *total currents* and *charges*, from the rigorous integral equation for the current and charge *densities*.

In the interior of the conductor let us assume a curve s defined as parallel, at every point, to the direction of the resultant current. We do not know precisely the path of this curve but we do know that such a curve can be drawn. In the case of wires of uniform cross section it will be approximately parallel to the axis of the wire. Let the cross section of the conductor normal to s be denoted by S. The total current I_s, parallel to s, is then given by

$$I_s = I = \int u_s dS.$$

Now corresponding to the surface S and its element dS, let us define a hypothetical surface $\sum$ and its element $d\sigma$ by the equation

$$u_s dS = I d\sigma,$$

whence

$$\int u_s dS = I = I\int d\sigma = I\cdot\sum,$$

so that $\sum$ is always unity. Now multiply the equation

$$\frac{1}{g}u_s = E_s^\circ - \frac{\partial}{\partial s}\Phi - i\omega A_s \tag{9}$$

by $d\sigma$ and integrate over the cross section $\sum$; we get

$$\int \frac{u_s}{g}d\sigma = \int Ed\sigma - i\omega\int A_s d\sigma - \frac{\partial}{\partial s}\int \Phi d\sigma.$$

This can be written as

$$r(s)I(s) = \bar{E}(s) - i\omega\bar{A}_s(s) - \frac{\partial}{\partial s}\bar{\Phi}(s),$$

or simply

$$rI = \bar{E} - i\omega\bar{A} - \frac{\partial}{\partial s}\bar{\Phi}; \tag{10}$$

r is simply the resistance per unit length of the conductor, since

$$rI^2 = \int \frac{u_s^2}{g}dS = \text{dissipation per unit length due to current } I_s,$$

while $\bar{E}$ is the mean impressed electric force, parallel to s, averaged over the surface $\sum$.

Now consider the term $i\omega\bar{A}$; we have

$$\bar{A} = \int A_s d\sigma = \int d\sigma\int\frac{u'_s}{r}dv, \qquad u' = u(t - r/c)$$

or, neglecting the retardation,

$$\bar{A} = \int d\sigma \int \frac{u_s}{r} dv.$$

We now assume that the " charging " current normal to s is negligibly small in its contribution to the vector potential, whence

$$\begin{aligned}\bar{A} &= \int ds' I(s') \cdot \cos(s, s') \int d\sigma \int d\sigma' \frac{1}{r} \\ &= \int I(s') \frac{\cos(s, s')}{r} \lambda(s, s') ds',\end{aligned}$$

where

$$\lambda(s, s') = \int d\sigma \int \frac{1}{r} d\sigma'.$$

The term $\bar{\Phi} = \int \Phi d\sigma$ of (10) is next to be considered. Writing

$$\rho dS = Q d\tau,$$

where Q is the total charge per unit length, it becomes

$$\int ds' Q(s') \int d\sigma \int \frac{1}{r} d\tau' = \int Q(s') \mu(s, s') ds',$$

and we get finally

$$rI = \bar{\bar{E}} - i\omega \int I \cdot \cos(s, s') \lambda(s, s') ds' - \frac{\partial}{\partial s} \int Q \cdot \mu(s, s') ds'. \quad (11)$$

This, together with the further relation

$$i\omega Q = -\frac{\partial}{\partial s} I, \quad (12)$$

constitutes an integral equation in the total current $I = I_s$. That is to say, we have succeeded in passing from the rigorous integral equation in the point function *densities* to an approximate integral equation in terms of the total current and charge per unit length of the conductor. The functions λ and μ of this equation, however, while theoretically determinable from the rigorous equation, are not solvable from the approximate integral equation. Indeed they are, strictly speaking, functions of the mode of distribution of the impressed field E°. This fact in most cases, however, is of purely academic interest and λ and μ can be approximately evaluated from the geometry of the conductor by assuming a certain distribution of current density over the cross section. With this problem, however, we have no concern here, we are merely concerned to deduce the form of the canonical equations of circuit theory.

Now let us integrate with respect to s, around the closed curve; we get

$$\begin{aligned}\int rIds &= \int \bar{E}ds - i\omega \int Ids \int \cos(s, s')\lambda(s, s')ds' \\ &= V - i\omega \int lIds,\end{aligned} \tag{13}$$

thus defining the *impressed voltage* V, and the inductance per unit length l. Finally, if we assume that this current variation along the conductor is negligibly small, we get

$$I \int rds = V - i\omega I \int lds,$$

which may be written as

$$RI + i\omega LI = V, \tag{14}$$

which is the usual form of the equation of circuit theory for a closed loop.

In deducing (14) from (10) there is one important point which should be noticed. The assumption that the variation in the current I along the conductor is sufficiently small to justify passing from (13) to (14) does not by any means imply that the effect of the distributed charge, which is absent in (14), is negligible. The term $(\partial/\partial s)\Phi$ vanishes in passing from (12) to (13) because the integration is carried around a closed path. Actually comparing the terms $i\omega\bar{A}$ and $(\partial/\partial s)\bar{\Phi}$, we see that their ratio involves the factor $(\omega/c)^2$ which is an exceedingly small quantity even at very high frequencies. Consequently extremely small variations in the current are sufficient to establish charges which can and do profoundly modify the resultant electric field. These, in the case of a closed circuit, are eliminated from explicit consideration by integrating around a closed curve.

This may be illustrated by brief consideration of a second case where the conductor is not closed but is terminated in the plates of a condenser at $s = s_1$ and $s = s_2$ respectively. Making the same assumption as above, after integrating (11) from $s = s_1$ to $s = s_2$, we get

$$RI + i\omega LI + \Phi_2 - \Phi_1 = V, \tag{15}$$

where $\Phi_2 - \Phi_1$ is the difference in Φ between the condenser plates. Assuming these very close together, $\Phi_2 - \Phi_1$ is approximately proportional to the charge on the condenser, that is, to

$$\int Idt = \frac{1}{i\omega} I,$$

and may be written as $I/\omega C$, whence

$$RI + i\omega LI + \frac{1}{i\omega C}I = V, \tag{16}$$

which is the usual circuit equation for series resistance, inductance and capacity.

Extension of the foregoing to networks containing a plurality of circuits or meshes is straightforward and involves no conceptual or physical difficulties, although branch points may be analytically troublesome. These questions will not be taken up, however, as the foregoing is sufficient to show the connection between general electromagnetic theory and circuit theory and to show how circuit equations may be rigorously derived and their limitations explicitly recognized.

The Telegraph Equation

A particularly interesting and instructive application of the preceding is to the problem of transmission along parallel wires and the assumptions underlying the engineering theory of transmission.[8]

Consider two equal and parallel straight wires so related to the impressed field that equal and opposite currents flow in the wires. Here, corresponding to equation (11), we have

$$\begin{aligned} rI = \bar{E} - i\omega \int I\{\lambda(s, s') - \lambda'(s, s')\}ds' \\ - \frac{\partial}{\partial s}\int Q\{\mu(s, s') - \mu'(s, s')\}ds'. \end{aligned} \tag{17}$$

In this equation $\lambda(s, s')$ is the "mutual inductance" between points s, s' in the same wire while $\lambda'(s, s')$ is the corresponding mutual inductance between point s in one wire and point s' in the other. μ and μ' have a corresponding significance as "mutual potential coefficients."

Now $\lambda(s, s') - \lambda'(s, s')$ is a rapidly decreasing monotonic function of $|s - s'|$ and the same statement holds for $\mu - \mu'$. In view of this property and further assuming the variation of I and Q with respect to s as small, (17) to a first approximation may be replaced by

$$rI = \bar{E} - i\omega I \int (\lambda - \lambda')ds' - \frac{\partial}{\partial s} Q \int (\mu - \mu')ds'. \tag{18}$$

At a sufficient distance from the physical terminals of the wires the

[8] For an entirely different treatment of this problem, reference may be made to "The Guided and Radiated Energy in Wire Transmission," *Trans. A. I. E. E.*, 1924.

integrals become independent of s and approach the limits

$$\int_{-\infty}^{\infty} (\lambda - \lambda')ds' = l,$$

$$\int_{-\infty}^{\infty} (\mu - \mu')ds' = \frac{1}{c},$$

whence

$$rI + i\omega lI + \frac{1}{i\omega c} I = \overline{E}.$$

Finally assuming that the impressed electric intensity $\overline{E} = 0$, and introducing the relation

$$i\omega Q = -\frac{\partial}{\partial s} I,$$

we get

$$\left(r + i\omega l + \frac{1}{i\omega c}\frac{\partial^2}{\partial s^2}\right) I = 0,$$

which is the *telegraph equation.*

Besides its formal theoretical interest the foregoing derivation of the telegraph equation admits of some deductions of practical importance. These deductions, which are rather obvious consequences of the analysis, may be listed as follows.

1. The telegraph equation, as derived above, applies with accuracy only at points at some distance from the physical terminals of the line.
2. The accuracy of the telegraph equation in formulating the physical phenomena decreases in general with increasing frequency.
3. The telegraph equation is the first approximate solution of an integral equation. The first approximate solution decreases in accuracy with decreasing wave length of the propagated current.
4. *While the telegraph equation indicates a finite velocity of propagation of the current along the line, it is based on the assumption that the fields of the currents and charges* (as derived from the potential functions Φ and A) *are propagated with infinite velocity.*
5. As a consequence of (4), the telegraph equation does not take into account the phenomena of *radiation,* and in fact indicates implicitly the absence of radiation.

The Coil Antenna

An important example of the type of problem to which the foregoing analysis is applicable is the coil antenna. To this problem

equations (11) and (12) immediately apply but, at least at high frequencies, the approximations introduced above to derive the telegraph equation are not legitimate. This is due to the geometry of the conductor, and also to the fact that the impressed field is not approximately concentrated but is distributed over the entire length of the coil. It is intended to apply these equations to a detailed study of this problem. In the meantime, however, it may be noted that the current depends *not only on the line integral of the impressed electric intensity but also on its mode of distribution along the length of the coil.* This fact may possibly have practical significance in the design of coil antenna and their calibration at very short wave lengths.

Appendix

In the beginning of this paper, it was stated that the analysis applied only to the case of conductors of unit permeability and specific inductive capacity which obey Ohm's Law. The reason for this restriction and the formal extension of the analysis to the more general case will now be briefly discussed.[9]

Suppose that the conductor, instead of having the restricted properties noted above, obeys Ohm's Law but has a permeability μ and specific inductive capacity k which may differ from unity.

The equation (1),

$$E = E^{\circ} - \text{grad}\,\Phi - i\omega A, \tag{1}$$

still holds, as do also the potential formulas (2) and (3) and the formulas for the electric and magnetic intensities (4) and (5). The relation

$$-iw\rho = \text{div}\,u$$

is also valid.

The equation $u = gE$ must, however, be modified in the following manner. If we write

$$P = \frac{k-1}{4\pi}E,$$

$$M = \frac{\mu-1}{4\pi\mu}H,$$

then the foregoing equations are correct, provided we substitute for the equation $u = gE$ the more general expression

$$u = gE + iwP + \text{curl}\,M.$$

[9] For a previous discussion, see "A Generalization of the Reciprocal Theorem," *B. S. T. J.*, July, 1924.

By aid of these relations, the problem involves the solution of the simultaneous integral equations

$$\boldsymbol{E} = \boldsymbol{E}^\circ - \text{grad}\,\Phi - iw\boldsymbol{A},$$

$$\boldsymbol{H} = \boldsymbol{H}^\circ + \text{curl}\,\boldsymbol{A}.$$

These simultaneous equations can immediately be reduced to a single integral equation in $\boldsymbol{u}$, the formal solution of which is straightforward. A study of this equation, however, has not been carried far enough to justify further discussion in the present paper.

Note on Vector Analysis and Notations

In the foregoing, vectors are indicated by Clarendon, or bold-faced type. To those unfamiliar with vector analysis the following may be helpful:

grad Φ is a vector with the Cartesian components

$$\text{grad}_x\,\Phi = \frac{\partial}{\partial x}\Phi, \qquad \text{grad}_y\,\Phi = \frac{\partial}{\partial y}\Phi, \qquad \text{grad}_z\,\Phi = \frac{\partial}{\partial z}\Phi;$$

curl $\boldsymbol{A}$ is a vector with the Cartesian components

$$\text{curl}_x\,\boldsymbol{A} = \frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y,$$

$$\text{curl}_y\,\boldsymbol{A} = \frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z,$$

$$\text{curl}_z\,\boldsymbol{A} = \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x;$$

div $\boldsymbol{u}$ is a scalar; in Cartesian notation

$$\text{div}\,\boldsymbol{u} = \frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y + \frac{\partial}{\partial z}u_z.$$

$(\boldsymbol{E}\cdot\boldsymbol{u})$ denotes the *scalar product* of the vectors $\boldsymbol{E}$ and $\boldsymbol{u}$ and itself is a scalar. In Cartesian notation

$$(\boldsymbol{E}\cdot\boldsymbol{u}) = E_x u_x + E_y u_y + E_z u_z.$$

$[\boldsymbol{E}\cdot\boldsymbol{H}]$ denotes the *vector product* of the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$. It is

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